

SINGULARITIES OF LOG VARIETIES VIA JET SCHEMES

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ABSTRACT. We study logarithmic jet schemes of a log scheme and generalize a theorem of M. Mustařa from the case of ordinary jet schemes to the logarithmic case. If X is a normal local complete intersection log variety, then X has canonical singularities if and only if the log jet schemes of X are irreducible.

1. INTRODUCTION

M. Mustařa proved in [9] that a normal, local complete intersection variety X has canonical singularities if and only if the jet schemes $J_m(X)$ are irreducible for all m . The goal of this article is to generalize this result to the case of log varieties.

We follow K. Kato's terminology (cf. [5]) and consider fine and saturated log schemes $\mathbf{X} = (X, \mathcal{M}_X)$, where X is a scheme and \mathcal{M}_X a sheaf of monoids giving the log structure on X . We call \mathbf{X} a log variety if X is a variety. The simplest log structures on X are associated to open embeddings $U \subset X$, with

$$\mathcal{M}_X = \{f \in \mathcal{O}_X \mid f \text{ is invertible on } U\}.$$

Special cases of this include the trivial log structure $U = X$, toric varieties $T \subset Y$, and toroidal embeddings $U_X \subset X$.

The log jet scheme $\mathbf{J}_m(\mathbf{X})$ of the log scheme \mathbf{X} was constructed in full generality by S. Dutter in [1]. (The special case where the log structure on X is associated to an open embedding $U \subset X$ was also worked out in [11].) The log jet scheme $\mathbf{J}_m(\mathbf{X})$ is again a log scheme. Its \mathbf{S} -valued points are the \mathbf{S} -valued log jets in \mathbf{X} :

$$\mathrm{Hom}_{\mathrm{log.sch}}(\mathbf{S}, \mathbf{J}_m(\mathbf{X})) \simeq \mathrm{Hom}_{\mathrm{log.sch}}(\mathbf{S} \times \mathbf{j}_m, \mathbf{X}).$$

Here \mathbf{S} is any log scheme, $j_m = \mathrm{Spec} k[t]/(t^{m+1})$ with trivial log structure $\mathcal{M}_{j_m} = \mathcal{O}_{j_m}^*$. This isomorphism is functorial in \mathbf{S} and gives the functor of points in $\mathbf{J}_m(\mathbf{X})$.

We are mostly interested in the underlying scheme $J_m(\mathbf{X})$ of the log jet scheme. The main theorem we prove is the following:

Theorem 1.1. *Let \mathbf{X} be a normal, local complete intersection log variety. Then \mathbf{X} has canonical singularities if and only if $J_m(\mathbf{X})$ is irreducible for all $m > 0$.*

We need to explain the notion of a normal, complete intersection log variety and when such a variety has canonical singularities. A log variety \mathbf{X} is a local complete intersection log variety if locally it is a complete intersection in a toric variety Y , with the log structure \mathcal{M}_X being the restriction of the standard log structure \mathcal{M}_Y on Y . We also require that X intersect the torus orbits of Y in proper dimension. The log variety \mathbf{X} is normal if the underlying variety X is normal.

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To define canonical singularities, we need the canonical divisor $K_{\mathbf{X}}$ of a log variety \mathbf{X} . This is defined more generally below, but for complete intersections in a toric variety it can be described as follows. Let $T \subset Y$ be a toric variety with its standard log structure. Then $K_{\mathbf{Y}} := K_Y + D_Y$, where K_Y is the canonical divisor of Y and D_Y is the reduced divisor $Y \setminus T$. The divisor $K_{\mathbf{Y}}$ is Cartier (in fact, it is trivial). If \mathbf{X} is a normal, complete intersection in \mathbf{Y} , we define $D_X = D_Y|_X$ and the canonical divisor $K_{\mathbf{X}} = K_X + D_X$. By adjunction, this divisor is again Cartier.

Now consider a morphism of log varieties $f : \tilde{\mathbf{X}} \rightarrow \mathbf{X}$, such that the underlying morphism of varieties $f : \tilde{X} \rightarrow X$ is proper birational. The discrepancy divisor of f is defined as the difference $K_{\tilde{\mathbf{X}}} - f^*(K_{\mathbf{X}})$. The discrepancy divisor is required to equal $D_{\tilde{X}} - f^*(D_X)$ away from the exceptional locus. We say that the log variety \mathbf{X} has canonical singularities if the discrepancy divisor is effective for every proper birational log morphism f .

We will show that the log variety \mathbf{X} is canonical if and only if the pair (X, D_X) is log canonical near D_X and $X \setminus D_X$ is canonical. This implies that to determine if \mathbf{X} is canonical, it suffices to consider one resolution $f : \tilde{\mathbf{X}} \rightarrow \mathbf{X}$. Similarly, we show that the irreducibility of the log jet scheme $J_m(\mathbf{X})$ can be expressed in terms of irreducibility and dimension of ordinary jet schemes. This reduces the proof of Theorem 1.1 to the cases considered in [9, 10].

In the theorem we require \mathbf{X} to be a normal log variety. In particular, all toric varieties are assumed to be normal. This is used in the proof to simplify the treatment of Cartier divisors on X . In Kato's theory of log schemes [5], normality does not play a large role. For example, log smooth varieties are usually not normal. It may be possible to extend the theorem to the non-normal case.

The requirement of the theorem that \mathbf{X} is a local complete intersection is essential. Similarly to the case of ordinary jets, it implies for example that the jet scheme $J_m(\mathbf{X})$ has no components of small dimension.

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2. LOG VARIETIES

We work over an algebraically closed field k of characteristic zero. The scheme-theoretic notation follows [3] and for log schemes we refer to [5]. All sheaves are taken in the Zariski topology.

2.1. Basics about log varieties. A semi-group is a set with a binary, associative operation. A *monoid* is a unitary semi-group; it has a unique identity element. We only consider commutative monoids, and every homomorphism of monoids is required to preserve the identity. Let $P = (P, +)$ denote a monoid throughout. Let $X = (X, \mathcal{O}_X)$ be a scheme. Under multiplication, \mathcal{O}_X forms a sheaf of (commutative) monoids. A *pre-log structure* (*pre-log str*) on X is the supplemental data of a sheaf \mathcal{M}_X of (commutative) monoids on X and a homomorphism of sheaves of monoids $\alpha_X : \mathcal{M}_X \rightarrow \mathcal{O}_X$. We usually refer only to “the pre-log str \mathcal{M}_X ,” suppressing α_X , and to α where no ambiguity arises. A morphism of pre-log structures is a morphism of sheaves of monoids $\mathcal{M}_X \rightarrow \mathcal{M}'_X$ compatible with the maps to \mathcal{O}_X .

A *pre-log scheme* is a scheme endowed with a pre-log str. Pre-log schemes are denoted in the form $\mathbf{X} = (X, \mathcal{M}_X)$. A pre-log scheme \mathbf{X} such that X is a variety is called a *pre-log variety*.

Let $\mathbf{X} = (X, \mathcal{M}_X)$ and $\mathbf{Y} = (Y, \mathcal{M}_Y)$ be pre-log schemes. A morphism (or *log morphism*) from \mathbf{X} to \mathbf{Y} is a pair of data given by a morphism $f : X \rightarrow Y$ and a morphism of pre-log structures $f^\flat : \mathcal{M}_Y \rightarrow f_*\mathcal{M}_X$. Such a morphism is denoted by (f, f^\flat) or simply f .

A pre-log str \mathcal{M}_X on X is called a *log structure (log str)* on X if restricting α induces an isomorphism $\alpha^{-1}\mathcal{O}_X^* \xrightarrow{\sim} \mathcal{O}_X^*$. A *log scheme* is a scheme endowed with a log str. If \mathbf{X} is both a pre-log variety and a log scheme, we call \mathbf{X} a *log variety*. A morphism of log schemes is a morphism of the underlying pre-log schemes.

Associated to any pre-log str on X is a natural log str \mathcal{M}_X^a on X formed by the pushout of the homomorphisms $\alpha^{-1}\mathcal{O}_X^* \rightarrow \mathcal{O}_X^*$ and $\alpha^{-1}\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X$:

$$\begin{array}{ccc} \mathcal{M}_X^a & \longleftarrow & \mathcal{M}_X \\ \uparrow & & \uparrow \\ \mathcal{O}_X^* & \longleftarrow & \alpha^{-1}\mathcal{O}_X^* \end{array}$$

The log str \mathcal{M}_X^a is called the *log str associated to \mathcal{M}_X* . This log str is universal for homomorphisms from \mathcal{M}_X to log structures on X .

Example 2.1. The *trivial* log str on X is defined by $\mathcal{M}_X = \mathcal{O}_X^*$ and the inclusion morphism $\alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X$. This log str is initial in the category of all log structures on X . For any log scheme \mathbf{Y} , a morphism of log schemes $\mathbf{Y} \rightarrow \mathbf{X}$ is equivalent to a morphism of schemes $Y \rightarrow X$.

Similarly, the log structure given by $\mathcal{M}_X = \mathcal{O}_X$ and the identity morphism $\alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X$ is final in the category of all log structures on X . In this case, for any log scheme \mathbf{Y} , a morphism of log schemes $\mathbf{X} \rightarrow \mathbf{Y}$ is equivalent to a morphism of schemes $X \rightarrow Y$.

Let X be any scheme, \mathbf{Y} a pre-log scheme, and $f : X \rightarrow Y$ a morphism of schemes. The inverse image sheaf $f^{-1}\mathcal{M}_Y$ together with the composition $f^{-1}\mathcal{M}_Y \rightarrow f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ defines a pre-log str on X . Further, if \mathbf{Y} is a log scheme, the *inverse image log str* is defined to be $(f^{-1}\mathcal{M}_Y)^a$ and is denoted $f^*\mathcal{M}_Y$. The inverse image log str is well-behaved; Kato notes that if \mathbf{Y} is a pre-log scheme, then $(f^{-1}\mathcal{M}_Y)^a \simeq f^*\mathcal{M}_Y^a$ (cf. [5]). When $f : X \rightarrow Y$ is an open or closed embedding, we may denote $f^*\mathcal{M}_Y$ by $\mathcal{M}_Y|_X$.

On the other hand, if \mathbf{X} is a log scheme, Y is any scheme, and $f : X \rightarrow Y$ is a morphism of schemes, then the *direct image log str* is the fibre product of the morphisms $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ and $f_*\mathcal{M}_X \rightarrow f_*\mathcal{O}_X$ (with the obvious structure morphism), denoted by $f_*^{log}\mathcal{M}_X$.

Example 2.2. Let $j : U \hookrightarrow X$, be an open embedding, give U the trivial log str $\mathcal{M}_U = \mathcal{O}_U^*$ and let $\mathcal{M}_X = j_*^{log}\mathcal{M}_U$. The local sections of \mathcal{M}_X are all the regular functions that are invertible on U . We call this \mathcal{M}_X the *log str associated to the open embedding $U \subset X$* .

In practice we only work with certain well-behaved (pre-)log structures. We will need some notions about monoids in order to introduce them.

Every monoid P contains its *group of units* P^* ; the quotient monoid P/P^* is denoted \overline{P} . Concretely, \overline{P} is the set of cosets $\{p + P^* : p \in P\}$ with the natural operation.

To every monoid P there is associated a group P^{gp} , the *groupification* of P , defined by the obvious universal property for homomorphisms from P to arbitrary abelian groups. Let p, p' and n, n' denote elements in P . The group P^{gp} is concretely described as the quotient of the product (of monoids) $P \times P$ by the equivalence

$$(p, n) \sim (p', n') \iff \exists m \in P : p + n' + m = p' + n + m.$$

If every equality $p + n = p + n'$ in P implies $n = n'$, then P is called *integral*. Alternatively, P is integral if the natural homomorphism $P \rightarrow P^{gp}$ taking $p \mapsto (p, 0)$ is injective. A finitely generated and integral monoid is called *fine*. P is called *saturated* if for any $p \in P^{gp}$ such that $mp = p + p + \dots + p$ lies in P for some integer $m > 0$, we have that $p \in P$.

These notions carry over to sheaves of monoids. Let the subsheaf of units of a sheaf of monoids \mathcal{M}_X on X be denoted \mathcal{M}_X^* . We also denote the sheaf of monoids $\mathcal{M}_X/\mathcal{M}_X^*$ by $\overline{\mathcal{M}}_X$, and the groupification of \mathcal{M}_X by \mathcal{M}_X^{gp} .

A monoid P and a homomorphism of monoids $P \rightarrow \mathcal{O}_X(X)$ determines the constant sheaf of monoids P_X on X , the pre-log str $\alpha : P_X \rightarrow \mathcal{O}_X$, and the associated log str $P_X^a \rightarrow \mathcal{O}_X$. A log structure \mathcal{M}_X is *quasi-coherent* if at any point of X there is a monoid P such that locally \mathcal{M}_X is isomorphic to P_X^a . More precisely, for any point $x \in X$, the pair of data comprised of an open neighbourhood of x , say $j : U \hookrightarrow X$, and a monoid P with a homomorphism $P \rightarrow \mathcal{O}_X(U)$ together defines a *chart of \mathcal{M}_X at x* if $j^*\mathcal{M}_X \simeq P_U^a$ as log structures on U . A *quasi-coherent* log scheme carries a quasi-coherent log str, or equivalently, admits a chart at every point. A log scheme is *coherent* (resp. *integral*, *fine*, *saturated*) if every point admits a chart (U, P) with finitely generated (resp. integral, fine, saturated) P . We are mostly interested in fine and saturated log varieties. If the log structure \mathcal{M}_X is fine and saturated, then every point $x \in X$ has a chart (U, P) such that the natural map induces an isomorphism $P \simeq \overline{\mathcal{M}}_{X,x}$ (Lemma 1.6 in [6]).

Example 2.3. Let X be a variety, $D \subset X$ a divisor and $V = X \setminus D$. The log str on X associated to the open embedding $V \subset X$ is fine and saturated. To construct a chart (U, P) at $x \in X$, let P be the monoid of effective Cartier divisors near x that are supported on D . This is an integral, finitely generated and saturated monoid. Choose local equations for elements in a basis of P^{gp} , defined on an open neighborhood U of x . Monomials in these local equations will give local equations of elements of P and hence a morphism of monoids $P \rightarrow \mathcal{O}_X(U)$.

2.2. The monoid algebra. Given a monoid P , the *monoid algebra* $k[P]$ is the set of finite formal sums $\{\sum a_p \chi^p : p \in P, a_p \in k\}$ with formal addition, and multiplication induced by the monoid operation (namely $\chi^p \cdot \chi^{p'} = \chi^{p+p'}$). One may consider the constant sheaf P_X on the space $X = \text{Spec } k[P]$; it yields a natural pre-log str on X arising from the canonical homomorphism $P \hookrightarrow k[P]$. Clearly the log scheme $\mathbf{X} = (\text{Spec } k[P], P_X^a)$ is quasi-coherent, and is integral, fine, or saturated if and only

if the monoid P is integral, fine or saturated. We call P_X^a the *standard* log str on the scheme $\operatorname{Spec} k[P]$. When P is a fine and saturated monoid, then $X = \operatorname{Spec} k[P]$ is an affine toric variety (cf. [2]). The standard log str on X is the same as the log str associated to the open embedding of the big torus $T \subset X$.

Let (X, \mathcal{M}_X) be a log scheme. To give a chart (U, P) at $x \in X$ is the same as to give a morphism of log schemes $(U, \mathcal{M}_X|_U) \rightarrow (Y = \operatorname{Spec} k[P], P_Y^a)$, such that the pullback of P_Y^a is isomorphic to $\mathcal{M}_X|_U$. If the log str \mathcal{M}_X is fine and saturated, we can choose P to be fine and saturated. Moreover, if X is of finite type, we can make P bigger if necessary (by adding units from \mathcal{O}_U^*) and assume that $U \rightarrow \operatorname{Spec} k[P]$ is a closed embedding. Thus, a fine and saturated log variety is locally a closed subvariety of a toric variety, with the log str \mathcal{M}_X being the restriction of the standard log str on the toric variety.

Generalizing the previous argument slightly, let $Y = \operatorname{Spec} R$ be an affine scheme with log str associated to the morphism $P \rightarrow R$ for a monoid P . If \mathbf{X} is any log scheme, then a morphism of log schemes $\mathbf{X} \rightarrow \mathbf{Y}$ is determined by a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_X(X) & \longleftarrow & P \\ \downarrow & & \downarrow \\ \mathcal{O}_X(X) & \longleftarrow & R. \end{array}$$

If $R = k[P]$ is the monoid algebra of P and Y has the standard log str, then the diagram is determined solely by the homomorphism of monoids $P \rightarrow \mathcal{M}_X(X)$.

2.3. Stratification of log varieties. Let $\mathbf{X} = (X, \mathcal{M}_X)$ be a fine and saturated log variety, $\overline{\mathcal{M}}_{X,x}$ the stalk of the sheaf of monoids $\overline{\mathcal{M}}_X$ at a point $x \in X$, and $(\overline{\mathcal{M}}_{X,x})^{gp}$ its associated group. This is a finitely generated free abelian group; let $r(x)$ be its rank. The function $r : X \rightarrow \mathbb{Z}_{\geq 0}$ divides X into a disjoint union:

$$X = \cup_i X_i, \quad X_i = r^{-1}(i).$$

Each locally closed set X_i can be given the structure of a locally closed subscheme of X as follows. Near a point $x \in X_i$ the ideal of X_i in \mathcal{O}_X is generated by $\alpha(\mathcal{M}_X \setminus \mathcal{M}_X^\times)$.

Example 2.4. When Y is a toric variety with its standard log str, the stratum Y_i is the union of codimension i torus orbits with reduced scheme structure. A chart (U, P) of a fine and saturated log scheme \mathbf{X} defines a morphism of schemes $U \rightarrow Y = \operatorname{Spec} k[P]$. The strata in U are the inverse images of the strata in Y .

Assumption 2.5. We will assume that X_i has codimension i in X for every i .

The codimension of X_i cannot be greater than i . If the codimension is less than i , but X_0 is non-empty, then the log jet schemes of \mathbf{X} are reducible (see Example 3.3 below). With some extra work it may be possible to define the canonical divisor $K_{\mathbf{X}}$ for such varieties and prove that they do not have canonical singularities. For simplicity, we will require the assumption.

2.4. Local complete intersection log varieties. Recall that a fine and saturated log variety $\mathbf{X} = (X, \mathcal{M}_X)$ is locally a closed subset of a toric variety $Y = \operatorname{Spec} k[P]$, such that the log str \mathcal{M}_X is the restriction of the standard log str on Y . We say that \mathbf{X} is a local complete intersection log variety if the closed embedding can be chosen to be a complete intersection. More precisely:

Definition 2.6. A log variety \mathbf{X} is a *local complete intersection* log variety if it satisfies Assumption 2.5 and every point $x \in X$ has a chart (U, P) with P a fine and saturated monoid such that $U \rightarrow Y = \operatorname{Spec} k[P]$ is a closed embedding whose image is a complete intersection in Y .

Note that a local complete intersection log variety is by definition fine and saturated. Assumption 2.5 means that the closed set $U \subset Y$ intersects all torus orbits in proper dimension.

Example 2.7. Let X be a non-singular variety, $D \subset X$ a reduced divisor. Let the log str \mathcal{M}_X be associated to the open embedding $X \setminus D \subset X$. Then \mathbf{X} is a local complete intersection log variety. Near $x \in X$, let f_1, \dots, f_m define the components of D . Then locally X is isomorphic to the graph of $f_1 \times \dots \times f_m$ in $Y = X \times \operatorname{Spec} k[z_1, \dots, z_m]$. Give Y the log str associated to the embedding $\{z_1 \cdots z_m \neq 0\} \subset Y$. Then X is a complete intersection in Y and the log str on X is the restriction of the log str on Y . The variety Y is not in general toric. However, since Y is non-singular, locally Y admits an étale morphism to a toric variety Z , with log str on Y pulled back from Z . This means that locally Y can be embedded as a hypersurface in the toric variety $Z \times k^*$ so that the morphism to Z is the projection.

2.5. Log varieties with canonical singularities. We say that a log variety \mathbf{X} is normal if the underlying variety X is normal. Consider normal, fine and saturated log varieties satisfying Assumption 2.5.

The *canonical divisor* of a log variety \mathbf{X} is

$$K_{\mathbf{X}} = K_X + D_X,$$

where K_X is the canonical divisor of the variety X and D_X is an effective Weil divisor defined as follows. Let $x \in X$ be the generic point of an irreducible divisor D . If the ideal generated by $\alpha(\mathcal{M}_X \setminus \mathcal{M}_X^\times)$ vanishes to order a at x , then D appears with coefficient a in D_X .

Since \mathbf{X} satisfies Assumption 2.5, the support of the divisor D_X is the closure of the stratum X_1 . The coefficient of D in D_X is equal to the length of the scheme X_1 at the generic point of D .

If the log str \mathcal{M}_X is associated to an open embedding $U \subset X$, then $D_X = X \setminus U$ is a reduced divisor. In particular, if Y is a toric variety with its standard log str, then $D_Y = Y \setminus T$ and hence $K_{\mathbf{Y}} = 0$. From the adjunction formula we get that $K_{\mathbf{X}}$ is a Cartier divisor for any local complete intersection log variety \mathbf{X} .

A morphism of log varieties $f : \mathbf{Y} \rightarrow \mathbf{X}$ is called *proper birational* if the underlying morphism of varieties $f : Y \rightarrow X$ is proper birational. Assume that $K_{\mathbf{X}}$ is Cartier, f proper birational, and let

$$(2.1) \quad K_{\mathbf{Y}} = f^* K_{\mathbf{X}} + \sum a_i E_i,$$

where E_i are irreducible divisors on Y ; to make the coefficients a_i unique, we require that if E_i is not exceptional, then a_i is the coefficient of E_i in $D_Y - f^*(D_X)$. The log variety \mathbf{X} has *canonical singularities* (or simply, is *canonical*) if the *total discrepancy divisor* $\sum a_i E_i$ is effective for every proper birational $f : \mathbf{Y} \rightarrow \mathbf{X}$.

Let us recall singularities of pairs (cf. [8]). A pair (X, D) consists of a normal variety X and a divisor D in X , such that $K_X + D$ is Cartier. For a proper birational morphism $Y \rightarrow X$, let

$$(2.2) \quad K_Y = f^*(K_X + D) + \sum b_i E_i,$$

where E_i are irreducible divisors on Y . One assumes that if E_i is not exceptional, then b_i is the coefficient of E_i in $-f^*(D)$. The divisor $\sum b_i E_i$ is called the total discrepancy divisor of f . The pair (X, D) is log canonical (resp. canonical) if the total discrepancy divisor has coefficients $b_i \geq -1$ (resp. $b_i \geq 0$). When $D = 0$, then the variety X is canonical if $b_i \geq 0$ for all i .

Proposition 2.8. *A log variety \mathbf{X} is canonical if and only if the pair (X, D_X) is log canonical and $X \setminus D_X$ is canonical.*

Proof. Let $\mathbf{Y} = (Y, \mathcal{M}_Y)$ be a log variety. Define $\mathbf{Y}' = (Y, \mathcal{M}'_Y)$, where \mathcal{M}'_Y is the log str associated to the open embedding $Y \setminus D_Y \subset Y$. Then the image of $\alpha : \mathcal{M}_Y \rightarrow \mathcal{O}_Y$ lies in the submonoid $\mathcal{M}_{Y'} \subset \mathcal{O}_Y$, hence the identity morphism of Y extends uniquely to a log morphism $\mathbf{Y}' \rightarrow \mathbf{Y}$. If $f : \mathbf{Y} \rightarrow \mathbf{X}$ is a proper birational morphism, we can compose this with $\mathbf{Y}' \rightarrow \mathbf{Y}$. After such a composition, the discrepancy divisor will not increase. Thus, to check if \mathbf{X} is canonical, it suffices to consider proper birational morphisms from \mathbf{Y} , where the log str \mathcal{M}_Y is associated to an open embedding $U \subset Y$. By the same argument, it suffices to take $U = Y \setminus f^{-1}(D_X)$. In this case $D_Y = f^{-1}(D_X)$ with reduced structure. Now comparing the discrepancy divisors for \mathbf{X} and (X, D_X) as in the formulas (2.1) and (2.2) above, we see that $a_i = b_i + 1$ if E_i maps to D_X , and $a_i = b_i$ otherwise. This implies the statement of the proposition. \square

As in the case of ordinary varieties, having canonical singularities can be checked on one resolution. We say that $f : (Y, D_Y) \rightarrow (X, D_X)$ is a log resolution of the pair (X, D_X) if Y is non-singular, $f^{-1}(D_X)$ is a divisor, $D_Y = f^{-1}(D_X)^{\text{red}}$ is the divisor with reduced structure, and D_Y together with the exceptional locus of f is a divisor of simple normal crossings.

Corollary 2.9. *Let \mathbf{X} be a log variety such that $K_{\mathbf{X}}$ is Cartier, and $f : (Y, D_Y) \rightarrow (X, D_X)$ a log resolution. Then \mathbf{X} has canonical singularities if and only if the divisor*

$$K_Y + D_Y - f^*(K_X + D_X)$$

is effective. \square

3. LOG JET SCHEMES

Let \mathbf{j}_m be the scheme $j_m = \text{Spec } k[t]/(t^{m+1})$ with its trivial log str. For any log scheme \mathbf{S} the product $\mathbf{S} \times \mathbf{j}_m$ exists in the category of schemes with log structures, as described in [5]. Concretely, since j_m is a one point space, S and $S \times j_m$ are homeomorphic. Since j_m has the trivial log str, the log str on $S \times j_m$ is the log str

associated to the pre-log str $\alpha_S : \mathcal{M}_S \rightarrow \mathcal{O}_{S \times j_m}$. This is defined by the pushout diagram:

$$\begin{array}{ccc} \mathcal{M}_{S \times j_m} & \longleftarrow & \mathcal{M}_S \\ \uparrow & & \uparrow \\ \mathcal{O}_{S \times j_m}^* & \longleftarrow & \alpha_S^{-1} \mathcal{O}_{S \times j_m}^*. \end{array}$$

Note that $\alpha_S^{-1} \mathcal{O}_{S \times j_m}^* = \mathcal{O}_S^*$, and

$$\mathcal{O}_{S \times j_m}^* = \mathcal{O}_S^* \times \mathcal{G},$$

where \mathcal{G} is the sheaf of multiplicative groups

$$\mathcal{G} = \{1 + a_1 t + \cdots a_m t^m : a_i \in \mathcal{O}_S\}.$$

It follows that $\mathcal{M}_{S \times j_m}$ is the product of sheaves $\mathcal{M}_S \times \mathcal{G}$, with $\alpha : \mathcal{M}_S \times \mathcal{G} \rightarrow \mathcal{O}_{S \times j_m}$ mapping

$$\alpha : (m, 1 + a_1 t + \cdots a_m t^m) \mapsto \alpha_S(m) \cdot (1 + a_1 t + \cdots a_m t^m).$$

Let \mathbf{X} be a fine and saturated log scheme. The m^{th} log jet scheme of \mathbf{X} , which we denote by $\mathbf{J}_m(\mathbf{X}) = (J_m(\mathbf{X}), \mathcal{M}_{J_m(\mathbf{X})})$, represents the functor

$$\mathbf{S} \mapsto \text{Hom}_{\text{log.sch}}(\mathbf{S} \times \mathbf{j}_m, \mathbf{X})$$

from the category of log schemes to that of sets. The existence of log jet schemes is proved in [1].

It is well-known that if $X \subset \mathbb{A}^n$ is a closed subscheme defined by l equations, then the jet scheme $J_m(X) \subset J_m(\mathbb{A}^n) \simeq \mathbb{A}^{(m+1)n}$ is a closed subscheme defined by $(m+1)l$ equations. In the next subsections we first recall the local equations for the ordinary jet scheme and then generalize this to the case of log jet schemes.

3.1. Equations defining the ordinary jet scheme. Let $Y = \mathbb{A}^n = \text{Spec } k[x_1, \dots, x_n]$. Then

$$J_m(Y) = \text{Spec } k[x_i^{(j)}]_{i=1, \dots, n; j=0, \dots, m}.$$

Let S be a scheme, and $R = \mathcal{O}_S(S)$. A morphism $S \rightarrow J_m(Y)$ given by $x_i^{(j)} \mapsto r_i^{(j)} \in R$ corresponds to the S -valued jet $S \times j_m \rightarrow Y$ given by

$$(3.3) \quad x_i \mapsto \sum_j r_i^{(j)} \frac{t^j}{j!}.$$

The coordinate ring of $J_m(Y)$ has a k -linear differential operator $d : k[x_i^{(j)}] \rightarrow k[x_i^{(j)}]$ that maps $x_i^{(j)} \mapsto x_i^{(j+1)}$, $x_i^{(m)} \mapsto 0$, and satisfies the Leibniz rule. If $f(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$, then

$$(3.4) \quad f\left(\sum_j x_1^{(j)} \frac{t^j}{j!}, \dots, \sum_j x_n^{(j)} \frac{t^j}{j!}\right) = \sum_j d^j f \frac{t^j}{j!}.$$

It follows from this that if $X \subset Y$ is the closed subscheme $X = V(f_1, \dots, f_l)$, then $J_m(X) \subset J_m(Y)$ is also a closed subscheme defined by

$$J_m(X) = V(d^j f_i)_{i=1, \dots, l; j=0, \dots, m}.$$

For later use we mention that the operator d extends to the ring $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}][x_i^{(j)}]$ and formula (3.4) also holds for Laurent polynomials $f \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

3.2. Equations defining the log jet scheme. Now let $P \subset P^{gp} \simeq \mathbb{Z}^n$ be a fine, saturated monoid, and $Y = \operatorname{Spec} k[P]$ a toric variety with its standard log str. Let $e_1, \dots, e_n \in P$ be a basis for P^{gp} , and let $x_1, \dots, x_n \in k[P]$ be the corresponding monomials. Note that elements of $k[P]$ are then Laurent polynomials in x_1, \dots, x_n .

We claim that the log jet scheme of \mathbf{Y} is

$$J_m(\mathbf{Y}) = \operatorname{Spec} k[P][\frac{x_i^{(j)}}{x_i}]_{i=1, \dots, n; j=1, \dots, m},$$

with log str induced by $P \hookrightarrow k[P]$. A morphism of log schemes $\mathbf{S} \rightarrow \mathbf{J}_m(\mathbf{Y})$ is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_S(S) & \longleftarrow & P \\ \downarrow & & \downarrow \\ \mathcal{O}_S(S) & \longleftarrow & k[P][\frac{x_i^{(j)}}{x_i}]. \end{array}$$

It is determined by a monoid homomorphism $P \rightarrow \mathcal{M}_S(S)$ and elements $\frac{x_i^{(j)}}{x_i} \mapsto \frac{r_i^{(j)}}{r_i} \in R = \mathcal{O}_S(S)$. (Note that we view $\frac{x_i^{(j)}}{x_i}$ as variables and $\frac{r_i^{(j)}}{r_i}$ as some elements in R , not necessarily quotients of two elements.)

An \mathbf{S} -valued jet in \mathbf{Y} is a morphism $\mathbf{S} \times \mathbf{j}_m \rightarrow \mathbf{Y}$, which corresponds to a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{S \times j_m}(S \times j_m) & \longleftarrow & P \\ \downarrow & & \downarrow \\ R[t]/(t^{m+1}) & \longleftarrow & k[P]. \end{array}$$

This is determined by a homomorphism of monoids $P \rightarrow \mathcal{M}_{S \times j_m}(S \times j_m)$. The monoid $\mathcal{M}_{S \times j_m}(S \times j_m)$ is the product of the monoid $\mathcal{M}_S(S)$ with the group

$$G = \{1 + a_1 t + \dots + a_m t^m \mid a_i \in R\} \subset (R[t]/(t^{m+1}))^*.$$

Hence the diagram above is determined by a homomorphism of monoids $P \rightarrow \mathcal{M}_S(S)$ and a homomorphism of groups $P^{gp} \rightarrow G$. Let the group homomorphism be given by

$$e_i \mapsto 1 + \sum_j \frac{r_i^{(j)}}{r_i} \frac{t^j}{j!}.$$

This gives a bijection between \mathbf{S} -valued points of the log jet scheme and \mathbf{S} -valued jets in \mathbf{Y} . One can easily check that this is an isomorphism of functors.

Suppose we have a morphism $\mathbf{S} \rightarrow \mathbf{J}_m(\mathbf{Y})$, given by $\phi : P \rightarrow \mathcal{M}_S(S)$ and $\frac{x_i^{(j)}}{x_i} \mapsto \frac{r_i^{(j)}}{r_i} \in R$. Set $r_i = \alpha(\phi(e_i)) \in R$, and $r_i^{(j)} = r_i \cdot \frac{r_i^{(j)}}{r_i}$. Then the ordinary S -valued jet into Y underlying the \mathbf{S} -valued jet into \mathbf{Y} is given by a k -algebra homomorphism

$k[P] \rightarrow R[t]/(t^{m+1})$. It maps $x_i \in P$ to

$$x_i \mapsto r_i(1 + \sum_{j>0} \frac{r_i^{(j)}}{r_i} \frac{t^j}{j!}) = r_i + \sum_{j>0} r_i^{(j)} \frac{t^j}{j!}.$$

This formula can be compared with (3.3) and it is the reason for denoting the coordinates on the log jet scheme by $\frac{x_i^{(j)}}{x_i}$.

Later, in the proof of the main theorem, we will need to consider birational morphisms of toric varieties and the induced morphisms of log jet schemes. Let $P \subset Q$ be two fine and saturated monoids, such that $P^{gp} = Q^{gp}$. The inclusion of monoids gives a morphism of toric varieties $f : Y' = \operatorname{Spec} k[Q] \rightarrow Y = \operatorname{Spec} k[P]$. This map is birational. With notation as above, we choose a basis $e_1, \dots, e_n \in P$ for P^{gp} and denote by x_1, \dots, x_n the corresponding monomials in both $k[P]$ and $k[Q]$. Then

$$J_m(\mathbf{Y}') = \operatorname{Spec} k[Q][\frac{x_i^{(j)}}{x_i}], \quad J_m(\mathbf{Y}) = \operatorname{Spec} k[P][\frac{x_i^{(j)}}{x_i}],$$

and one easily checks that the induced morphism of log jet schemes $J_m(\mathbf{Y}') \rightarrow J_m(\mathbf{Y})$ takes $\frac{x_i^{(j)}}{x_i}$ to $\frac{x_i^{(j)}}{x_i}$. This result can be restated by saying that the diagram

$$\begin{array}{ccc} J_m(\mathbf{Y}') & \longrightarrow & J_m(\mathbf{Y}) \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

is a fibre square. In fact, the diagram is also a fibre square in the category of log schemes when we give each scheme its log structure.

To study the log jet schemes of closed subschemes of Y , we need to define the k -linear derivation $d : k[P][\frac{x_i^{(j)}}{x_i}] \rightarrow k[P][\frac{x_i^{(j)}}{x_i}]$. Note that we have an inclusion of rings

$$k[P][\frac{x_i^{(j)}}{x_i}] \subset k[x_1^{\pm 1}, \dots, x_n^{\pm 1}][x_i^{(j)}].$$

The derivation d on $k[x_1, \dots, x_n][x_i^{(j)}]$ defined in the case of ordinary jet schemes extends to a derivation on the ring $k[x_1^{\pm 1}, \dots, x_n^{\pm 1}][x_i^{(j)}]$. The subring $k[P][\frac{x_i^{(j)}}{x_i}]$ is invariant by this extension, so we define the desired derivation d as the restriction of the extended d . Invariance of the subring under d follows from:

$$\begin{aligned} d(\prod_i x_i^{a_i}) &= (\prod_i x_i^{a_i}) (\sum_i a_i \frac{x_i^{(1)}}{x_i}), \\ d(\frac{x_i^{(j)}}{x_i}) &= \frac{x_i^{(j+1)} x_i - x_i^{(1)} x_i^{(j)}}{x_i^2} = \frac{x_i^{(j+1)}}{x_i} - \frac{x_i^{(1)}}{x_i} \frac{x_i^{(j)}}{x_i}. \end{aligned}$$

Now let $X \subset Y$ be a closed subscheme defined by $X = V(f_1, \dots, f_l)$. Let the log str on X be the restriction of the log str on Y .

Lemma 3.1. *The log jet scheme $J_m(\mathbf{X})$ is a closed subscheme of $J_m(\mathbf{Y})$, defined by*

$$J_m(\mathbf{X}) = V(d^j f_i)_{i=1,\dots,l; j=0,\dots,m}.$$

The log str on $J_m(\mathbf{X})$ is pulled back from $J_m(\mathbf{Y})$.

Proof. The proof is analogous to the proof in the ordinary jet case. An \mathbf{S} -valued log jet in \mathbf{X} is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_S(S) \times G & \longleftarrow & P \\ \downarrow & & \downarrow \\ R[t]/(t^{m+1}) & \longleftarrow & k[P]/(f_1, \dots, f_l), \end{array}$$

where $R = \mathcal{O}_S(S)$ and G is as above. This is determined by a monoid homomorphism $P \rightarrow \mathcal{M}_S(S) \times G$ such that the resulting ring homomorphism $\phi : k[P] \rightarrow R[t]/(t^{m+1})$ takes f_1, \dots, f_l to zero. Recall that ϕ maps

$$x_i \mapsto r_i + \sum_{j>0} r_i^{(j)} \frac{t^j}{j!}.$$

Since any $f \in k[P]$ is a Laurent polynomial in variables x_1, \dots, x_n , it follows that ϕ takes f to

$$f(x_1^{\pm 1}, \dots, x_n^{\pm 1}) \mapsto \sum_j d^j f \frac{t^j}{j!} \Big|_{x_i=r_i, \frac{x_i^{(j)}}{x_i} = \frac{r_i^{(j)}}{r_i}}.$$

To make sense of a monomial $\chi^p \in k[P] \subset k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ evaluated at $x_i = r_i$, we have to define it as the image of p under $P \rightarrow \mathcal{M}_S(S) \rightarrow R$. Therefore, f maps to zero if and only if all $d^j f$ when evaluated at $r_i, \frac{r_i^{(j)}}{r_i}$ are zero.

The statement of the lemma now follows. \square

The lemma implies that if \mathbf{X} is a local complete intersection log variety, then every component of $J_m(\mathbf{X})$ has dimension at least $(m+1) \dim X$.

3.3. Log jets in terms of ordinary jets. Let \mathbf{X} be a fine and saturated log variety. Recall that we have a stratification of X by locally closed subschemes $X = \cup_j X_j$. Let $J_m(\mathbf{X}) = \cup_j J_m(\mathbf{X})_j$, where $J_m(\mathbf{X})_j$ is the inverse image of X_j under the projection $J_m(\mathbf{X}) \rightarrow X$. (Note that $J_m(\mathbf{X})_j$ could also stand for the j -th stratum in the stratification of $J_m(\mathbf{X})$ as a log scheme. However, these two constructions agree because the log str on $J_m(\mathbf{X})$ is the pullback of the log str on X .)

Lemma 3.2. *$J_m(\mathbf{X})_j$ is a locally trivial \mathbb{A}^{mj} -bundle over $J_m(X_j)$.*

Proof. Fix a point $x \in X_j$ and a chart (U, P) at x , such that $U = \text{Spec } A$ and $P \simeq \overline{\mathcal{M}}_{X,x}$. We replace X by U .

To study morphisms of schemes $S \rightarrow J_m(\mathbf{X})$, we give S the log str $\mathcal{M}_S = \mathcal{O}_S$ and consider morphisms of log schemes $\mathbf{S} \rightarrow \mathbf{J}_m(\mathbf{X})$, or equivalently, \mathbf{S} -valued log jets in

X. These correspond to commutative diagrams

$$\begin{array}{ccc} R \times G & \longleftarrow & P \\ \downarrow & & \downarrow \\ R[t]/(t^{m+1}) & \longleftarrow & A, \end{array}$$

where $R = \mathcal{O}_S(S)$ and $G = \{1 + a_1 t + \cdots + a_m t^m \mid a_i \in R\}$ is as above. The stratum X_j near x is defined by the vanishing of all non-units in P . For the jet to lie in $J_m(\mathbf{X})_j$, we need that every $p \in P \setminus \{0\}$ maps to an element of the form $\sum_{i \geq 0} r_i t^i$. This means that the monoid homomorphism $P \rightarrow R \rightarrow R[t]/(t^{m+1})$ maps $P \setminus \{0\}$ to zero. Then also $P \rightarrow R \times G \rightarrow R[t]/(t^{m+1})$ maps $P \setminus \{0\}$ to zero. In other words, the ideal in A defining X_j maps to zero, hence the underlying S -valued jet in X is a jet in X_j . This gives a morphism $J_m(\mathbf{X})_j \rightarrow J_m(X_j)$.

Given an S -valued jet in X_j , the extra data needed to get an \mathbf{S} -valued jet in \mathbf{X} is a monoid homomorphism $P \rightarrow G$, equivalently a group homomorphism $P^{gp} \simeq \mathbb{Z}^j \rightarrow G$. This implies that $J_m(\mathbf{X})_j \simeq J_m(X_j) \times \mathbb{A}^{mj}$. \square

Example 3.3. Suppose \mathbf{X} does not satisfy Assumption 2.5 and some X_j has codimension less than j . Then for m large we have an inequality

$$\dim J_m(\mathbf{X})_j = \dim J_m(X_j) + mj \geq (m+1) \dim X_j + mj > (m+1) \dim X.$$

Thus, if X_0 is non-empty, then the dimension of $J_m(\mathbf{X})_j$ is greater than the dimension of $J_m(\mathbf{X})$ over some non-empty open locus in X , hence the log jet scheme is reducible.

4. A LEMMA ABOUT LOG CANONICAL PAIRS

We consider here pairs of the form $(X, \alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_r Z_r)$, where Z_1, \dots, Z_r are closed subschemes in a normal \mathbb{Q} -Gorenstein variety X and $\alpha_1, \dots, \alpha_r$ are non-negative real numbers. Let $f : Y \rightarrow X$ be a log resolution of the pair, meaning that f is a proper birational morphism, Y is non-singular, the scheme theoretic inverse images $f^{-1}(Z_i)$ are Cartier divisors, and the union of these inverse images together with the exceptional locus of f forms a divisor of simple normal crossings. Then the pair is called log canonical if the total discrepancy divisor of f :

$$Tot.discr = K_{Y/X} - \sum_i \alpha_i f^{-1}(Z_i)$$

has coefficients ≥ -1 . The log canonical threshold $\text{lct}(X, \alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_r Z_r)$ is the maximal $\alpha > 0$, such that $(X, \alpha(\alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_r Z_r))$ is log canonical. In particular, the pair is log canonical if and only if its log canonical threshold is ≥ 1 .

The multiplier ideal sheaf of the pair $(X, \alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_r Z_r)$ is, with notation as above,

$$\mathcal{J}(X, \alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_r Z_r) = f_* \mathcal{O}_Y(\lceil Tot.discr \rceil).$$

Note that the multiplier ideal sheaf is equal to \mathcal{O}_X if and only if the coefficients of the total discrepancy divisor are > -1 . The pair is log canonical if and only if $\mathcal{J}(X, (1 - \epsilon)(\alpha_1 Z_1 + \alpha_2 Z_2 + \cdots + \alpha_r Z_r)) = \mathcal{O}_X$ for any $\epsilon > 0$.

For a point $x \in X$, the log canonical threshold of the pair $(X, \alpha_1 Z_1 + \cdots + \alpha_r Z_r)$ at x is

$$\text{lct}_x(X, \alpha_1 Z_1 + \cdots + \alpha_r Z_r) = \text{lct}(U, \alpha_1 Z_1|_U + \cdots + \alpha_r Z_r|_U),$$

where U is a sufficiently small open neighborhood of x . We say that the pair is log canonical at x if the log canonical threshold at x is ≥ 1 . This is equivalent to the equality of stalks

$$\mathcal{J}(X, (1 - \epsilon)(\alpha_1 Z_1 + \cdots + \alpha_r Z_r))_x = \mathcal{O}_{X,x}$$

for any $\epsilon > 0$.

More generally, for a closed subset $C \subset X$, we say that the pair $(X, \alpha_1 Z_1 + \cdots + \alpha_r Z_r)$ is log canonical near C if it is log canonical in some open neighborhood of C . Equivalently, the pair is log canonical at every point $x \in C$.

Lemma 4.1. *Let X be a non-singular variety and Z_1, \dots, Z_r irreducible closed subschemes of codimension c_1, \dots, c_r , respectively. Then the pair*

$$(X, (c_1 + \cdots + c_r)(Z_1 \cap \cdots \cap Z_r))$$

is log canonical if and only if the pair

$$(X, c_1 Z_1 + \cdots + c_r Z_r)$$

is log canonical near $Z_1 \cap \cdots \cap Z_r$.

Proof. This proof is due to M. Mustaș.

Let $c = c_1 + \cdots + c_r$.

\Leftarrow . Assume that $(X, c_1 Z_1 + \cdots + c_r Z_r)$ is log canonical. Let a_i be the ideal sheaf of Z_i , $1 \leq i \leq r$. Since $a_1^{c_1} \cdots a_r^{c_r} \subset (a_1 + \cdots + a_r)^c$, monotonicity of log canonical threshold implies

$$\text{lct}(X, c(Z_1 \cap \cdots \cap Z_r)) \geq \text{lct}(X, c_1 Z_1 + \cdots + c_r Z_r) \geq 1,$$

and hence $(X, c(Z_1 \cap \cdots \cap Z_r))$ is also log canonical.

\Rightarrow . Let $(X, c(Z_1 \cap \cdots \cap Z_r))$ be log canonical, but assume by contradiction that $(X, c_1 Z_1 + \cdots + c_r Z_r)$ is not log canonical at some point $x \in Z_1 \cap \cdots \cap Z_r$. Consider

$$\phi(\lambda_1, \dots, \lambda_r) = \text{lct}_x(X, \lambda_1 Z_1 + \cdots + \lambda_r Z_r)$$

as a function of $\lambda_1, \dots, \lambda_r$ in the domain $\lambda_i \geq 0$, $\sum_i \lambda_i = c$. If some $\lambda_i > c_i$, then $\phi(\lambda_1, \dots, \lambda_r) < 1$ because the pair $(X, \lambda_i Z_i)$ is not log canonical in any neighborhood of x . By assumption, $\phi(c_1, \dots, c_r) < 1$, hence $\phi(\lambda_1, \dots, \lambda_r) < 1$ for all $(\lambda_1, \dots, \lambda_r)$ in the compact domain. Since the log canonical threshold ϕ depends continuously on λ_i , we can find $\epsilon > 0$, such that

$$\phi(\lambda_1, \dots, \lambda_r) < 1 - \epsilon$$

for all $\lambda_i \geq 0$, $\sum_i \lambda_i = c$.

A theorem of Jow and Miller [4] states that the multiplier ideal sheaves satisfy

$$\mathcal{J}(X, \alpha(Z_1 \cap \cdots \cap Z_r)) = \sum_{\lambda_1 + \cdots + \lambda_r = \alpha} \mathcal{J}(X, \lambda_1 Z_1 + \cdots + \lambda_r Z_r).$$

Since $(X, c(Z_1 \cap \cdots \cap Z_r))$ is log canonical, the left hand side is equal to \mathcal{O}_X for any $\alpha < c$. Let us fix $\alpha = c - \eta$ for some $\eta > 0$. Then on the right hand side for some $\lambda_i \geq 0$, $\lambda_1 + \cdots + \lambda_r = c - \eta$,

$$\mathcal{J}(X, \lambda_1 Z_1 + \cdots + \lambda_r Z_r)_x = \mathcal{O}_{X,x},$$

hence

$$\text{lct}_x(X, \lambda_1 Z_1 + \cdots + \lambda_r Z_r) \geq 1.$$

Scaling this by $\frac{c}{c-\eta}$, we get

$$\text{lct}_x(X, \frac{c}{c-\eta}(\lambda_1 Z_1 + \cdots + \lambda_r Z_r)) \geq \frac{c-\eta}{c}.$$

However, $\sum_i \frac{\lambda_i c}{c-\eta} = c$, and hence

$$\text{lct}_x(X, \frac{c}{c-\eta}(\lambda_1 Z_1 + \cdots + \lambda_r Z_r)) < 1 - \epsilon.$$

Combining the two inequalities, we get

$$\frac{c-\eta}{c} < 1 - \epsilon$$

for any $\eta > 0$. Taking $\eta < c\epsilon$ gives a contradiction. \square

We stated the lemma in the form that will be used below. It can be proved more generally. The variety X needs to be normal, \mathbb{Q} -Gorenstein, Z_i closed subschemes and the numbers c_i must satisfy

$$c_i \geq \text{lct}(X, Z_i)$$

near $Z_1 \cap \cdots \cap Z_r$. Then the same conclusions hold.

5. PROOF OF THE MAIN THEOREM

We now prove Theorem 1.1. The problem is local, so we assume that $Y = \text{Spec } k[P]$ is a toric variety and $X \subset Y$ is a complete intersection of codimension c , and that X is normal and intersects the torus orbits in Y in proper dimension. The log str on Y is the standard one and it induces the log str on X . In particular, $D_X = D_Y|_X$.

We need to prove that $J_m(\mathbf{X})$ is irreducible for all m if and only if (X, D_X) is log canonical near D_X and $X \setminus D_X$ is canonical. Note that $X \setminus D_X$ is a complete intersection in the torus $T = Y \setminus D_Y$ and the log jet scheme $J_m(\mathbf{X})$ over $X \setminus D_X$ is the ordinary jet scheme $J_m(X \setminus D_X)$. By Mustař's theorem for local complete intersection varieties [9], $X \setminus D_X$ has canonical singularities if and only if $J_m(X \setminus D_X)$ is irreducible for all m . We will assume these equivalent conditions in the following. Thus, we need to show that $J_m(\mathbf{X})$ has no components lying over D_X if and only if (X, D_X) is log canonical.

5.1. The case of non-singular Y . Assume that Y is a non-singular toric variety. For later use we will relax the condition of X being normal. We consider complete intersections X that are non-singular in codimension 1, but could be non-normal at some points in D_X . The notion of (X, D_X) being log canonical is the same as before.

Lemma 5.1. *The pair (X, D_X) is log canonical if and only if the pair $(Y, cX + D_Y)$ is log canonical.*

Proof. The proof is the same as the corresponding reduction in [9]. Let $f : \tilde{Y} \rightarrow Y$ be the blowup of Y along X , with F the exceptional divisor. Then F is a locally trivial \mathbb{P}^{c-1} -bundle over X . Hence (X, D_X) is log canonical if and only if $(F, f^*(D_X))$ is log canonical. Notice that F is also non-singular in codimension 1.

The log canonicity of $(Y, cX + D_Y)$ can be checked on a resolution that factors through \tilde{Y} . The blowup \tilde{Y} is a local complete intersection, hence Gorenstein. Since $F \subset \tilde{Y}$ is non-singular in codimension 1, so is \tilde{Y} . Then from $K_{\tilde{Y}/Y} = (c-1)F$ it follows that $(Y, cX + D_Y)$ is log canonical if and only if the pair $(\tilde{Y}, F + f^*D_Y)$ is log canonical.

Applying inverse of adjunction [7], the pair $(F, f^*D_X) = (F, f^*D_Y|_F)$ is log canonical if and only if the pair $(\tilde{Y}, F + f^*D_Y)$ is log canonical. \square

Recall that we have a stratification $X = \cup_{l \geq 0} X_l$ of X into locally closed subschemes, and the corresponding stratification of the log jet schemes $J_m(\mathbf{X}) = \cup_l J_m(\mathbf{X})_l$, where each $J_m(\mathbf{X})_l$ is a locally trivial \mathbb{A}^{ml} -bundle over $J_m(X_l)$. By Assumption 2.5, X_l has codimension l in X , hence $(c+l)$ is the codimension of X_l in Y .

To simplify notation, we will write (Y, X_l) for the pair $(Y \setminus \cup_{i>l} X_i, X_l)$.

Lemma 5.2. *The log jet scheme $J_m(\mathbf{X})$ is irreducible for any $m > 0$ if and only if the pair $(Y, (c+l)X_l)$ is log canonical for any $l > 0$.*

Proof. Recall that since X is a complete intersection, all components of $J_m(\mathbf{X})$ have dimension at least $(m+1)d$, where $d = \dim X$. The log jet scheme $J_m(\mathbf{X})$ has no components over D_X if and only if for any $l > 0$,

$$(5.5) \quad \dim J_m(\mathbf{X})_l = \dim J_m(X_l) + ml < d(m+1).$$

By the main theorem in [10], the pair $(Y, (c+l)X_l)$ is log canonical if and only if for any m

$$(5.6) \quad \dim J_m(X_l) \leq (m+1)(d-l).$$

Moreover,

$$\text{lct}(Y, X_l) = d + c - \frac{\dim J_m(X_l)}{m+1}$$

for any $m+1$ large and divisible enough.

If $(Y, (c+l)X_l)$ is log canonical for all l , then inequality (5.6) holds for all l . This implies that inequality (5.5) also holds for all l , hence $J_m(\mathbf{X})$ is irreducible.

Conversely, if $J_m(\mathbf{X})$ is irreducible for all m , then the inequality (5.5) holds for all m, l . Computing the log canonical threshold, we get

$$\text{lct}(Y, X_l) > d + c - \frac{(m+1)d - ml}{m+1} = c + l - \frac{l}{m+1},$$

for any $m+1$ large and divisible enough and for any $l > 0$. This implies that $\text{lct}(Y, X_l) \geq c + l$ and hence $(Y, (c+l)X_l)$ is log canonical for any $l > 0$. \square

Lemma 5.3. *The pair $(Y, (c+l)X_l)$ is log canonical if and only if $(Y, cX + D_Y)$ is log canonical near X_l .*

Proof. Let $x \in X_l$ and let D_1, \dots, D_l be the irreducible toric divisors in Y containing x . Then $X_l = X \cap D_1 \cap \dots \cap D_l$ and $D_Y = D_1 + \dots + D_l$ near x . Now apply Lemma 4.1 to the pair $(Y, cX + D_1 + \dots + D_l)$. \square

The last two lemmas show that $J_m(\mathbf{X})$ is irreducible if and only if $(Y, cX + D_Y)$ is log canonical near X_l for any l . This means that $J_m(\mathbf{X})$ is irreducible if and only if $(Y, cX + D_Y)$ is log canonical. Lemma 5.1 then finishes the proof of the main theorem in the case when Y is non-singular.

5.2. The case of arbitrary toric Y . Let $f : \tilde{Y} \rightarrow Y$ be a toric resolution of singularities of Y . Then $K_{\tilde{Y}} + D_{\tilde{Y}} = f^*(K_Y + D_Y)$ because both of these divisors are trivial.

Let $\tilde{X} = f^{-1}(X)$. Since X is a complete intersection and it intersects the torus orbits of Y properly, it follows that \tilde{X} is an irreducible, complete intersection in \tilde{Y} , and the morphism $f : \tilde{X} \rightarrow X$ is birational. Let \tilde{X} have the log str induced by the log str on \tilde{Y} .

Lemma 5.4. *$J_m(\mathbf{X})$ is irreducible if and only if $J_m(\tilde{\mathbf{X}})$ is irreducible.*

Proof. Let $y \in \tilde{Y}$. Since the log morphism $f : \tilde{\mathbf{Y}} \rightarrow \mathbf{Y}$ is birational and log smooth, the morphism of jets

$$J_m(\tilde{\mathbf{Y}})_y \rightarrow J_m(\mathbf{Y})_{f(y)}$$

is an isomorphism. Here $J_m(\tilde{\mathbf{Y}})_y$ is the fibre of $\pi_m : J_m(\tilde{\mathbf{Y}}) \rightarrow \tilde{Y}$ over y .

A log jet into \mathbf{X} is the same as a log jet into \mathbf{Y} , with image lying in X . This implies that if $y \in \tilde{X}$ then

$$J_m(\tilde{\mathbf{X}})_y \rightarrow J_m(\mathbf{X})_{f(y)}$$

is also an isomorphism.

If $J_m(\tilde{\mathbf{X}})$ is irreducible, so is $J_m(\mathbf{X})$ because it is the image of $J_m(\tilde{\mathbf{X}})$.

Conversely, suppose $J_m(\tilde{\mathbf{X}})$ is reducible. Then for some l ,

$$\dim J_m(\tilde{X}_l) \geq (m+1)d - ml.$$

Let $S \subset \tilde{Y}$ be a torus orbit of codimension l , such that

$$\dim J_m(\tilde{X} \cap S) \geq (m+1)d - ml.$$

Let $T \subset Y$ be the image of S . Then T is a torus orbit of codimension $j \geq l$. The morphism $f : S \rightarrow T$ has $(j-l)$ dimensional fibres. It follows that

$$\dim J_m(X \cap T) \geq (m+1)d - ml - (j-l) = (m+1)d - mj + [(m-1)(j-l)].$$

Since $(m-1)(j-l) \geq 0$,

$$\dim J_m(X_j) \geq (m+1)d - mj,$$

hence $J_m(\mathbf{X})$ is reducible. \square

Lemma 5.5. *The pair (X, D_X) is log canonical if and only if the pair $(\tilde{X}, D_{\tilde{X}})$ is log canonical.*

Proof. The morphism $f : \tilde{X} \rightarrow X$ is proper birational, hence it is enough to show that $f^*(K_X + D_X) = K_{\tilde{X}} + D_{\tilde{X}}$.

Since X is a complete intersection in Y , it follows that

$$K_X + D_X = (K_Y + D_Y)|_X.$$

Similarly,

$$\begin{aligned} K_{\tilde{X}} + D_{\tilde{X}} &= K_{\tilde{Y}} + D_{\tilde{Y}}|_{\tilde{X}} \\ &= f^*(K_Y + D_Y)|_{\tilde{X}} \\ &= f^*(K_X + D_X). \end{aligned}$$

\square

The two lemmas reduce the case of general Y to the case of smooth Y . Note that \tilde{X} may not be normal, but since f is an isomorphism in codimension 1, \tilde{X} is non-singular in codimension 1 and the non-normal points all lie in $D_{\tilde{X}}$.

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